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The Calculation of Some Limiting Distributions Arising in Near-Integrated Models with GLS Detrending

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Abstract

Many unit root test statisticss are nowadays constructed using detrended data, with the method of GLS detrending being popular in the setting of a near-integrated model. This paper determines the properties of some associated limiting distributions when the GLS detrending is based on a linear time trend. A fundamental result for the moment generating function of two key functionals of the relevant stochastic process is provided and used to compute probability density functions and cumulative distribution functions, as well as means and variances, of the limiting distributions of some statistics of interest. Some further applications, including a comparison of limiting power functions and the consideration of a more complicated statistic, are also provided.

Keywords. GLS detrending; near-integrated model; moment generating function; characteristic function.

J.E.L. classification numbers. C22; C46; C63.

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1. Introduction

An important issue in the conduct of tests for a unit root in a time series concerns the specification of the trend component. In recent years it has become common to detrend the series prior to carrying out the test regression, and generalised least squares (GLS) detrending (or quasi-differencing) has earned a prominent place in the literature. Elliott, Rothenberg and Stock (1996) and Ng and Perron (2001) analysed the properties of a number of statistics based on GLS-detrended data in a near-integrated model and demonstrated that the limiting distributions depend on the form of trend function employed for the detrending. When the data are detrended using only an intercept the limiting distributions are functionals of an Ornstein-Uhlenbeck (OU) process rather than a standard Wiener process, the parameter characterising the OU process being the parameter that measures the deviation from a unit root (sometimes called the local-to-unity coefficient). The moment generating function (MGF) and characteristic function (CF) of two functionals of the OU process were derived by Phillips (1987), and Perron (1989) used these functions to derive the cumulative distribution function (CDF) and probability density function (PDF) in a numerical study of the limiting distribution of the ordinary least squares (OLS) estimator in a first-order near-integrated autoregression. When both an intercept and a time trend are used in the GLS detrending the limiting distributions depend on a more general process that is a function of the GLS-detrending parameter as well as the local-to-unity coefficient and the OU process. To date, analytical results relating to certain functionals of this more complicated process, such as the MGF and CF, appear not to have been derived, and one of the aims of this paper is to fill this gap in the literature. Another aim is to use such an MGF or CF to analyse the properties of the limiting distributions of certain test statistics by, for example, deriving the CDF and PDF, in a similar way to Perron (1989).

The paper is organised as follows. Section 2 describes the GLS detrending procedure and defines the random processes (and functionals thereof) that are important in characterising the limiting distributions of certain test statistics, while Section 3 derives the MGF and CF of the two key random functionals. Sections 4 and 5 use the results of Section 3 to derive the moments, as well as the CDF and PDF, of two test statistics of interest, before Section 6 provides further discussion and concluding comments. There are also three appendices: Appendix A provides proofs of the theorems presented in the main text; Appendix B gives a supplementary result that is used in the proof of one of the theorems; and Appendix C provides computational details of the results presented in the text.

2. GLS detrending and asymptotics

A common theoretical framework for testing for a unit root in a time series, and one that is commonly applied in practice, is based on detrending the series of interest using a deterministic trend function prior to computing the test statistic using the detrended data. Although a variety of forms of deterministic trend could be envisaged it is usually specified to be a low-order polynomial and is typically linear in practice. To be precise suppose that the scalar random variable of interest, y_t, has the representation

$$y_t = d_t + u_t, \quad t = 1, \ldots, n, \tag{1}$$

where d_t denotes the deterministic trend and u_t is an unobservable scalar random process assumed to satisfy

$$u_t = \alpha u_{t-1} + v_t, \quad v_t = \delta(L)\epsilon_t, \quad \epsilon_t \sim iid(0, \sigma_\epsilon^2), \quad t = 1, \ldots, n, \tag{2}$$

where $\alpha = 1 + c/n$ for some constant c, $\delta(z) = \sum_{j=0}^{\infty} \delta_j z^j$, $\delta_0 = 1$, $\sum_{j=0}^{\infty} j|\delta_j| < \infty$ and L denotes the lag operator. This specification is consistent with v_t being a stationary ARMA(p,q) process of the form $\rho(L)v_t = \theta(L)\epsilon_t$ where $\rho(z) = \sum_{j=0}^{p} \rho_j z^j$ and $\theta(z) = \sum_{j=0}^{q} \theta_j z^j$, in which case $\delta(z) = \theta(z)/\rho(z)$, but it also allows for more general forms of linear processes. Under these assumptions v_t satisfies the functional central limit theorem $n^{-1/2} \sum_{t=1}^{[nr]} v_t \Rightarrow \sigma W(r)$ on $C[0,1]$, where $\sigma^2 = \sigma_\epsilon^2 \delta(1)^2$ denotes the long-run variance and $[nr]$ denotes the integer part of nr. The deterministic component, d_t, in (1) is usually assumed to be of the form $d_t = \psi' z_t$ where $z_t = [1, t, t^2, \ldots, t^p]'$, most interest focusing on the cases $p = 0$ and $p = 1$. Under (1) and (2) the detrended series $y_t - d_t$ satisfies

$$y_t - d_t = \alpha(y_{t-1} - d_{t-1}) + v_t,$$

hence the objective being to test the null hypothesis that $\alpha = 1$ or, equivalently, that $c = 0$. Note that when $c < 0$ the process is said to be locally stationary while when $c > 0$ it is locally explosive.

In view of d_t being unobservable a common procedure is to estimate ψ, using an estimator $\hat{\psi}$ (to be defined below), and to construct a detrended series of the form $y_t^d = y_t - \hat{\psi}' z_t$ to be used in place of $y_t - d_t$ above. The GLS procedure, proposed by Elliott, Rothenberg and Stock (1996), can be described as follows. Let $\bar{\alpha} = 1 + \bar{c}/n$ denote the detrending parameter, $\bar{c}$ being a suitably chosen constant, and, for any series $x_0, x_1, \ldots, x_n$, define the quasi-differenced variables $x_0^{\bar{\alpha}} = x_0$ and $x_t^{\bar{\alpha}} = x_t - \bar{\alpha} x_{t-1}$ $(t = 1, \ldots, n)$. Then $\hat{\psi}$ is obtained from the OLS regression of $y_t^{\bar{\alpha}}$ on $z_t^{\bar{\alpha}}$. Elliott, Rothenberg and Stock (1996) recommend that when $p = 0$, $\bar{c} = -7$ and when $p = 1$, $\bar{c} = -13.5$, these values being chosen so as to make the asymptotic local power function of tests tangent to the asymptotic Gaussian power envelope at the point where power equals one half.

The GLS-detrended series y_t^d can be used in the construction of a variety of test statistics. Elliott, Rothenberg and Stock (1996) proposed a feasible statistic, P_n, whose limiting distribution is the same as that of a likelihood-based point-optimal test statistic; it is defined by

$$P_n = \frac{S(\bar{\alpha}) - \bar{\alpha} S(1)}{\hat{\sigma}^2}, \tag{3}$$

where $\hat{\sigma}^2$ is a consistent estimator of the long run variance σ^2 and $S(\alpha)$ denotes the sum of squared residuals from a least squares regression of y_t^α on z_t^α for the values of α specified in (3). However, the most common approach in practice is based on either an estimate of the parameter α itself (or its equivalent in an alternative representation) or on its associated t-ratio. Nonparametric treatments of the serial correlation inherent in v_t can be conducted using the methods of Phillips and Perron (1988) based on the OLS regression

$$y_t^d = \tilde{\beta}_0 y_{t-1}^d + \tilde{v}_t, \quad t = 1, \ldots, n, \tag{4}$$

while parametric treatments are often based on an augmented Dickey-Fuller (ADF) regression of the form

$$y_t^d = \hat{\beta}_0 y_{t-1}^d + \sum_{j=1}^{k} \hat{\beta}_j \Delta y_{t-j}^d + \hat{e}_{tk}, \quad t = k+1, \dots, n, \tag{5}$$

where k can be chosen, for example, using the modified information criterion proposed by Ng and Perron (2001). The null hypothesis in either case corresponds to $\beta_0 = 1$ where β_0 is the coefficient on y_{t-1}^d. In (4) the limiting distribution of the normalised estimator $\tilde{\beta}_0$ depends on nuisance parameters emanating from the dynamics associated with v_t, but an asymptotically pivotal distribution can be obtained by conducting inference using $n(\tilde{\beta}_0 - 1) + k_n$, where k_n denotes the nonparametric data-based adjustment term; see Phillips and Perron (1988) for details. A similar type of nonparametric adjustment can be applied to the t-ratio based on $\tilde{\beta}_0$ in order to obtain a pivotal limiting distribution. These limiting distributions correspond to those that are obtained from the ADF regression (5) using $n(\hat{\beta}_0 - 1)$ and its t-ratio $t_0 = (\hat{\beta}_0 - 1)/\hat{\sigma}_{\hat{\beta}_0}$, where $\hat{\sigma}^2_{\hat{\beta}_0}$ denotes the OLS estimator of the variance of $\hat{\beta}_0$, provided that k is allowed to increase with n at a suitable rate. In order to subsequently save on notation the focus will be on $n(\hat{\beta}_0 - 1)$ and t_0 but it is emphasised at this point that the same properties of the limiting distributions also apply to $n(\tilde{\beta}_0 - 1) + k_n$ and the corresponding nonparametrically adjusted t-ratio from the regression (4). The properties of these limiting distributions, as well as those of P_n, are investigated in subsequent sections.

For the detrended variable y_t^d Elliott, Rothenberg and Stock (1996) established that

$$n^{-1/2} y_{[nr]}^d \Rightarrow \begin{cases} \sigma W_c(r), & p = 0, \\ \sigma V_{c,\bar{c}}(r), & p = 1, \end{cases}$$

where $W_c(r)$ and $V_{c,\bar{c}}(r)$ are random processes on $r \in [0,1]$ and the symbol $\Rightarrow$ denotes weak convergence of the relevant probability measures. In fact $W_c(r)$ is the Ornstein-Uhlenbeck process satisfying $dW_c(r) = cW_c(r)dr + dW(r)$ where $W(r)$ is a standard Wiener process with $W(0) = 0$, and therefore has the representations

$$W_c(r) = \int_0^r \exp\{c(r-s)\} dW(s) = W(r) + c \int_0^r \exp\{c(r-s)\} W(s) ds;$$

see Phillips (1987) for details. The process $V_{c,\bar{c}}(r)$, on the other hand, is more complicated and is given by

$$V_{c,\bar{c}}(r) = W_c(r) - r \left(\lambda W_c(1) + 3(1-\lambda) \int_0^1 s W_c(s) ds \right),$$

where $\lambda = (1-\bar{c})/(1-\bar{c}+\bar{c}^2/3)$.

The processes $W_c(r)$ and $V_{c,\bar{c}}(r)$ characterise the limiting distributions of the statistics of interest. These distributions can be expressed in terms of the following functionals of the underlying random processes:

$$N_c = \frac{1}{2}\left(W_c(1)^2 - 1 \right) = \int_0^1 W_c dW_c, \quad D_c = \int_0^1 W_c^2,$$

$$N_{c,\bar{c}} = \frac{1}{2}\left(V_{c,\bar{c}}(1)^2 - 1 \right) = \int_0^1 V_{c,\bar{c}} dV_{c,\bar{c}}, \quad D_{c,\bar{c}} = \int_0^1 V_{c,\bar{c}}^2, \tag{6}$$

in addition to $W_c(1)^2$ and $V_{c,\bar{c}}(1)^2$ themselves. When $p = 0$ the limiting distributions, as

$n \to \infty$, have the representations

$$P_n \Rightarrow \bar{c}^2 D_c - \bar{c} W_c(1)^2, \quad n(\hat{\beta}_0 - 1) \Rightarrow \frac{N_c}{D_c}, \quad t_0 \Rightarrow \frac{N_c}{D_c^{1/2}}, \tag{7}$$

while when $p = 1$ they take the form

$$P_n \Rightarrow \bar{c}^2 D_{c,\bar{c}} + (1 - \bar{c}) V_{c,\bar{c}}(1)^2, \quad n(\hat{\beta}_0 - 1) \Rightarrow \frac{N_{c,\bar{c}}}{D_{c,\bar{c}}}, \quad t_0 \Rightarrow \frac{N_{c,\bar{c}}}{D_{c,\bar{c}}^{1/2}}. \tag{8}$$

Note that the limiting distribution of $n(\hat{\beta}_0 - \beta_0)$ is obtained straightforwardly from the results in (7) and (8) by using the fact that $\beta_0 = 1 + c/n$ and hence $n(\hat{\beta}_0 - \beta_0) = n(\hat{\beta}_0 - 1) - c$. For example, when $p = 1$ it follows that

$$n(\hat{\beta}_0 - \beta_0) \Rightarrow \frac{N_{c,\bar{c}}}{D_{c,\bar{c}}} - c.$$

For the case $p = 0$ the joint MGF and CF of N_c and D_c were derived by Phillips (1987) and were used by Perron (1989) to analyse the properties of the CDF and PDF for different values of the parameter c using numerical integration techniques. Analagous results for the joint MGF and CF of $V_{c,\bar{c}}(1)^2$ (or $N_{c,\bar{c}}$) and $D_{c,\bar{c}}$ do not yet appear to have been derived and so the next section deals with this problem. The results are more complicated than when $p = 0$ owing to the fact that the process $V_{c,\bar{c}}(r)$ is itself a functional of $W_c(r)$.

3. The joint moment generating and characteristic functions of $V_{c,\bar{c}}(1)^2$ and $\int_0^1 V_{c,\bar{c}}(r)^2 dr$

All of the limiting distributions for $p = 1$ in the previous section are characterised by the random variables $V_{c,\bar{c}}(1)^2$ and $\int_0^1 V_{c,\bar{c}}(r)^2 dr$; their joint MGF is defined by

$$M(t_1, t_2) = E\left[\exp\left(t_1 V_{c,\bar{c}}(1)^2 + t_2 \int_0^1 V_{c,\bar{c}}(r)^2 dr\right)\right].$$

Although $M(t_1, t_2)$ is also a function of c and $\bar{c}$ this is not stated explicitly for reasons of notational economy. The CF is then obtained using the expression

$$\Phi(t_1, t_2) = E\left[\exp\left(it_1 V_{c,\bar{c}}(1)^2 + it_2 \int_0^1 V_{c,\bar{c}}(r)^2 dr\right)\right] = M(it_1, it_2),$$

where $i^2 = -1$. The precise form of $M(t_1, t_2)$ is given in Theorem 1 below.

Theorem 1. *The joint MGF of $V_{c,\bar{c}}(1)^2$ and $\int_0^1 V_{c,\bar{c}}(r)^2 dr$ is given by*

$$M(t_1, t_2) = \exp\left(-\frac{c}{2}\right) H(t_1, t_2)^{-1/2},$$

where

$$H(t_1, t_2) = h_1(t_1, t_2) \sinh \gamma + h_2(t_1, t_2) \cosh \gamma,$$

$\gamma = \sqrt{c^2 - 2t_2}$, *and $h_1(t_1, t_2)$ and $h_2(t_1, t_2)$ are functions of t_1 and t_2 of the form*

$$h_i = (-1)^i + \sum_{j=1}^{4} h_{ij} a_j(t_1, t_2), \quad i = 1, 2,$$

where

$$a_i(t_1,t_2) = a_{i0} + a_{i1}t_1 + a_{i2}t_2, \quad i = 1,2,3,$$

$$a_4(t_1,t_2) = a_1(t_1,t_2)a_3(t_1,t_2) - a_2(t_1,t_2)^2,$$

and the coefficients h_{ij} $(i = 1,2; j = 1,\ldots,4)$ *and* a_{ij} $(i = 1,2,3; j = 0,1,2)$ *are defined in Table 1.*

The CF is easily derived from the MGF in Theorem 1 and has the representation

$$\Phi(t_1,t_2) = \exp\left(-\frac{c}{2}\right) H(it_1,it_2)^{-1/2}.$$

The method used to derive the MGF in Theorem 1 is described as the "stochastic process approach" in Tanaka (1996) and involves a change of measure (using Girsanov's Theorem) allied with the normality of the underlying OU process to evaluate the expectation of interest. The separate MGFs of $V_{c,\bar{c}}(1)^2$ and $\int_0^1 V_{c,\bar{c}}(r)^2 dr$ follow straightforwardly (with some further algebra) from the joint MGF in Theorem 1.

Corollary to Theorem 1. *The MGF of* $V_{c,\bar{c}}(1)^2$ *is given by*

$$M_1(t_1) = M(t_1,0) = \left[1 + e^c\left(k_1 \sinh c + k_2 \cosh c\right) t_1\right]^{-1/2},$$

and the MGF of $\int_0^1 V_{c,\bar{c}}(r)^2 dr$ *is given by*

$$\begin{aligned} M_2(t_2) = M(0,t_2) \;=\; & e^{(\gamma-c)/2}\Big[1 + e^{\gamma}\Big(k_{10}\sinh\gamma + (k_{11}\sinh\gamma + k_{21}\cosh\gamma)\,t_2 \\ & + (k_{12}\sinh\gamma + k_{22}\cosh\gamma)\,t_2^2\Big)\Big]^{-1/2}, \end{aligned}$$

where $k_{10} = h_{11}a_{10}$*, the* k_i $(i = 1,2)$ *and* k_{ij} $(i,j = 1,2)$ *are of the form*

$$\begin{aligned} k_i &= h^c_{i1}a_{11} + h^c_{i2}a_{21} + h^c_{i3}a_{31}, \\ k_{i1} &= h_{i1}a_{12} + h_{i2}a_{22} + h_{i3}a_{32} + h_{i4}a_{10}a_{32}, \\ k_{i2} &= h_{i4}(a_{12}a_{32} - a_{22}^2), \end{aligned}$$

and the coefficients h^c_{ij} $(i = 1,2; j = 1,\ldots,3)$, h_{ij} $(i = 1,2; j = 1,\ldots,4)$ *and* a_{ij} $(i = 1,2,3; j = 0,1,2)$ *are defined in Table 1.*

Various uses of the MGF in Theorem 1 are described in the following sections.

4. The limiting distribution of P_n

The limiting distribution of P_n was given in (8) and can be represented by the random variable

$$S_{c,\bar{c}} = (1-\bar{c})V_{c,\bar{c}}(1)^2 + \bar{c}^2 \int_0^1 V_{c,\bar{c}}(r)^2 dr. \tag{9}$$

Let $m(t) = E\exp(tS_{c,\bar{c}})$ denote the MGF of $S_{c,\bar{c}}$. It follows from Theorem 1 that

$$m(t) = M\left((1-\bar{c})t, \bar{c}^2 t\right),$$

and the first two moments of $S_{c,\bar{c}}$ can be obtained using

$$E(S_{c,\bar{c}}) = \left.\frac{dm(t)}{dt}\right|_{t=0} = \left.\frac{dM((1-\bar{c})t, \bar{c}^2 t)}{dt}\right|_{t=0},$$

$$E(S_{c,\bar{c}}^2) = \left.\frac{d^2m(t)}{dt^2}\right|_{t=0} = \left.\frac{d^2M((1-\bar{c})t, \bar{c}^2 t)}{dt^2}\right|_{t=0}.$$

Precise expressions for these moments are presented in Theorem 2.

Theorem 2. *Let $S_{c,\bar{c}}$ be defined in (9). Then*

$$E(S_{c,\bar{c}}) = \begin{cases} -\frac{1}{2}e^{c}\left(\frac{\bar{c}^2}{c}e^{-c} + \eta_1 \sinh c + \eta_2 \cosh c\right), & (c \neq 0), \\ \frac{1}{30}\left(6(1-\bar{c})(1-\lambda)^2 + 2\bar{c}\lambda^2 + 3\bar{c}^2\right), & (c = 0), \end{cases}$$

$$E(S_{c,\bar{c}}^2) = \begin{cases} \frac{3}{4}e^{2c}\left(\frac{\bar{c}^2}{c}e^{-c} + \eta_1 \sinh c + \eta_2 \cosh c\right)^2 - \frac{1}{2}e^{c}\left(\frac{\bar{c}^4}{c^2}e^{-c}\left(1+\frac{1}{c}\right)\right) \\ -\frac{1}{2}e^{c}\left(\left(\eta_3 - 2\frac{\bar{c}^2}{c}\eta_2\right)\sinh c + \left(\eta_4 - 2\frac{\bar{c}^2}{c}\eta_1\right)\cosh c\right), & (c \neq 0), \\ \frac{\bar{c}^2}{315}\left(36(1-\bar{c})(1-\lambda)^2 + 56\bar{c}^2\lambda^4 + 12\bar{c}\lambda^2 + 9\bar{c}^2\right), & (c = 0), \end{cases}$$

where $\eta_i = \sum_{j=1}^{7} \eta_{ij} c^{-j}$ $(i = 1, \ldots, 4)$ and the η_{ij} coefficients are given in Table 2.

The mean and variance of $S_{c,\bar{c}}$ for a range of values of c from -20 to $+2$ are given in Table 3. Both the mean and variance rise as c approaches zero from below and then fall slightly before increasing rapidly when c exceeds unity and extends further into the explosive region.

The CF of $S_{c,\bar{c}}$, denoted $\phi(t) = E\exp(itS_{c,\bar{c}})$, is obtained from the MGF by replacing t with it in its definition, yielding

$$\phi(t) = m(it) = M\left((1-\bar{c})it, \bar{c}^2 it\right).$$

It can be used to derive the CDF, $F(z)$, and PDF, $f(z)$, of $S_{c,\bar{c}}$ using the following formulae:

$$F(z) = \frac{1}{2} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-itz}\phi(t)}{t}dt = \frac{1}{2} - \frac{1}{\pi}\int_{0}^{\infty} \frac{\text{Im}\{(e^{-itz}\phi(t)\}}{t}dt,$$

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itz}\phi(t)dt = \frac{1}{\pi}\int_{0}^{\infty} \text{Re}\{e^{-itz}\phi(t)\}dt,$$

where, for a complex-valued variable x, $\text{Re}\{x\}$ and $\text{Im}\{x\}$ denote the real and imaginary parts, respectively; the second expressions for $F(z)$ and $f(z)$ were used in the computations reported below.

The CDF and PDF of $S_{c,\bar{c}}$ are plotted in Figures 1 and 2, respectively, for values of $c \in \{-10, -5, 0, 2\}$. As c increases through this range of values the distribution can be seen to shift to the right and become more dispersed in accordance with the values for the mean and variance in Table 3. Selected percentage points for the same range of values of c as used in Table 3 are given in Table 4; these were computed using the bisection method described in Tanaka (1996, p.203). Of particular relevance are the values when $c = 0$ which are those that would be needed under the null hypothesis of a unit root using the statistic P_n. It is of interest to compare these values with those reported in Elliott, Rothenberg and Stock (1996) and Ng and Perron (2001) which were obtained by simulation. The 1%, 5% and 10% values – 3.9756, 5.6900 and 6.9853, respectively – obtained using the exact methods compare with 3.96, 5.62 and 6.89 obtained by the former authors and 4.03, 5.48 and 6.67 obtained by the latter. The simulation methods appear to understate the 5% and 10% critical values with the reported 1% values being closer to the exact value.

5. The limiting distributions of $n(\hat{\beta}_0 - 1)$ and $n(\hat{\beta}_0 - \beta_0)$

The limiting distribution of $n(\hat{\beta}_0 - 1)$ is characterised by the ratio $N_{c,\bar{c}}/D_{c,\bar{c}}$, and the joint MGF of the numerator and denominator, $Q(\theta_1, \theta_2)$, can be obtained straightforwardly from the MGF $M(t_1, t_2)$ given in Theorem 1, as follows:

$$\begin{aligned} Q(\theta_1, \theta_2) &= E\left[\exp\left(\theta_1 N_{c,\bar{c}} + \theta_2 D_{c,\bar{c}}\right)\right] \\ &= E\left[\exp\left(\frac{\theta_1}{2}\left(V_{c,\bar{c}}(1)^2 - 1\right) + \theta_2 \int_0^1 V_{c,\bar{c}}(r)^2 dr\right)\right] \\ &= \exp\left(-\frac{\theta_1}{2}\right) E\left[\exp\left(\frac{\theta_1}{2} V_{c,\bar{c}}(1)^2 + \theta_2 \int_0^1 V_{c,\bar{c}}(r)^2 dr\right)\right] \\ &= \exp\left(-\frac{\theta_1}{2}\right) M\left(\frac{\theta_1}{2}, \theta_2\right). \end{aligned}$$

The CF of $N_{c,\bar{c}}$ and $D_{c,\bar{c}}$ is then given by $\Psi(\theta_1, \theta_2) = Q(i\theta_1, i\theta_2)$. The moments of the ratio $N_{c,\bar{c}}/D_{c,\bar{c}}$ can then be obtained using

$$E\left(\frac{N_{c,\bar{c}}}{D_{c,\bar{c}}}\right)^k = \frac{1}{(k-1)!} \int_0^\infty \theta_2^{k-1} \left.\frac{\partial^k Q(\theta_1, -\theta_2)}{\partial \theta_1^k}\right|_{\theta_1 = 0} d\theta_2; \tag{10}$$

see, for example, Mehta and Swamy (1978) and Magnus (1986). Expressions for the first two moments are given below.

Theorem 3. *Let $N_{c,\bar{c}}$ and $D_{c,\bar{c}}$ be defined as in (6). Then the first two moments of the ratio $N_{c,\bar{c}}/D_{c,\bar{c}}$ are given by*

$$E\left(\frac{N_{c,\bar{c}}}{D_{c,\bar{c}}}\right) = -(I_1 + I_2), \quad E\left(\frac{N_{c,\bar{c}}}{D_{c,\bar{c}}}\right)^2 = I_3 + I_4 + I_5,$$

where

$$I_1 = \frac{1}{2} \exp\left(-\frac{c}{2}\right) \int_0^\infty \frac{1}{p(\theta_2)^{1/2}} d\theta_2,$$

$$I_2 = \frac{1}{2} \exp\left(-\frac{c}{2}\right) \int_0^\infty \frac{q(\theta_2)}{p(\theta_2)^{3/2}} d\theta_2,$$

$$I_3 = \frac{1}{2} \exp\left(-\frac{c}{2}\right) \int_0^\infty \theta_2 \frac{q(\theta_2)}{p(\theta_2)^{3/2}} d\theta_2,$$

$$I_4 = \frac{1}{4} \exp\left(-\frac{c}{2}\right) \int_0^\infty \theta_2 \frac{1}{p(\theta_2)^{1/2}} d\theta_2,$$

$$I_5 = \frac{3}{4} \exp\left(-\frac{c}{2}\right) \int_0^\infty \theta_2 \frac{q(\theta_2)^2}{p(\theta_2)^{5/2}} d\theta_2,$$

$$p(\theta_2) = p_1(\theta_2) \sinh \sqrt{c^2 + 2\theta_2} + p_2(\theta_2) \cosh \sqrt{c^2 + 2\theta_2},$$

$$q(\theta_2) = q_1(\theta_2) \sinh \sqrt{c^2 + 2\theta_2} + q_2(\theta_2) \cosh \sqrt{c^2 + 2\theta_2},$$

and for $i = 1, 2$,

$$p_i = (-1)^i + \sum_{j=1}^{4} h_{ij} a_j(0, -\theta_2), \quad q_i = \sum_{j=1}^{4} h_{ij} \left. \frac{\partial a_j(\theta_1, -\theta_2)}{\partial \theta_1} \right|_{\theta_1 = 0},$$

the h_{ij} and a_j coefficients being defined in Theorem 1 and Table 1.

The means and variances of $N_{c,\bar{c}}/D_{c,\bar{c}}$ are provided for a range of values of c in Table 5; in all cases $\bar{c} = -13.5$. It can be seen that the mean increases with c apart from a slight fall around $c = 0$ while the variance falls with c apart from a small increase around $c = 0$.

The CDF of the ratio $N_{c,\bar{c}}/D_{c,\bar{c}}$ can be obtained using a result of Gurland (1948), in view of $\Pr(D_{c,\bar{c}} \leq 0) = 0$, as follows:

$$\begin{aligned} G(z) = \lim_{n \to \infty} \Pr\left(n(\hat{\beta}_0 - 1) < z\right) &= \frac{1}{2} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Psi(\theta_1, -\theta_1 z)}{\theta_1} d\theta_1 \\ &= \frac{1}{2} - \frac{1}{\pi} \int_{0}^{\infty} \frac{\operatorname{Im}\{\Psi(\theta_1, -\theta_1 z)\}}{\theta_1} d\theta_1, \end{aligned} \quad (11)$$

$\Psi(\theta_1, \theta_2)$ being the CF of $N_{c,\bar{c}}$ and $D_{c,\bar{c}}$ defined earlier. Furthermore, the PDF can be obtained either by computing another integral of the form

$$g(z) = \frac{d}{dz} G(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left. \frac{\partial \Psi(\theta_1, \theta_2)}{\partial \theta_2} \right|_{\theta_2 = -\theta_1 z} d\theta_1,$$

or by numerically differentiating the CDF using

$$g(z) = \frac{G(z + h) - G(z)}{h}$$

for some small value of h; see Tanaka (1996, p.197). The latter approach avoids issues involved with the integration of a further derivative of the CF and follows straightforwardly once the integral in (11) can be computed. The results for the PDF reported below were obtained with $h = 10^{-6}$. Figures 3 and 4 depict the CDF and PDF, respectively, for the same range of values of c as were used in Figures 1 and 2. It can be seen that as c increases in value the distribution shifts to the right and becomes less dispersed which is in accordance with the computed values for the mean and variance in Table 5. In addition, selected percentage points of the distribution for a range of values of c are given in Table 6. Of particular relevance for unit root testing are the entries for $c = 0$ which could be used as critical values for testing for a unit root using the statistic $n(\hat{\beta}_0 - 1)$.

Results for the limiting distribution of $n(\hat{\beta}_0 - \beta_0)$ follow straightforwardly from the above results. As noted at the end of section 2

$$n(\hat{\beta}_0 - \beta_0) \Rightarrow \frac{N_{c,\bar{c}}}{D_{c,\bar{c}}} - c,$$

and hence the CDF, $G_0(z)$, can be obtained from $G(z)$ as follows:

$$G_0(z) = \lim_{n \to \infty} \Pr\left(n(\hat{\beta}_0 - \beta_0) < z\right) = \lim_{n \to \infty} \Pr\left(n(\hat{\beta}_0 - 1) - c < z\right) = G(z + c).$$

Similarly, the PDF, $g_0(z)$, satisfies $g_0(z) = g(z + c)$. The CDF and PDF are depicted in

Figures 5 and 6, respectively, for the same values of c as were used in Figures 1–4. The curves in Figures 5 and 6 are closer together than those in Figures 3 and 4 owing to the horizontal translation by an amount equal to $-c$.

6. Discussion and concluding comments

The results presented in the preceding sections are potentially of use whenever certain calculations concerning the limiting distributions are required. One such possible application is in the comparison of the asymptotic power of the two statistics, P_n and $n(\hat{\beta}_0 - 1)$, in testing for a unit root. For example, in the case of P_n, let z_α denote the α-percentage point of the limiting distribution when $c = 0$ i.e. under the null hypothesis of a unit root; when $\alpha = 0.05$ this value can be seen from Table 4 to be $z_{0.05} = 5.69$. Then the power of the size-α test for testing the null against stationary alternatives is given by computing $F(z_\alpha)$ for $c < 0$ using the expression in section 4; a similar procedure for the statistic $n(\hat{\beta}_0 - 1)$ can be followed using the expression for the CDF in section 5 allied with the critical value obtained from Table 6. The results of such a power comparision of P_n and $n(\hat{\beta}_0 - 1)$ are given in Table 7 for values of α corresponding to 1%, 5% and 10% level tests. Both tests have broadly the same power although $n(\hat{\beta}_0 - 1)$ tends to have slightly higher power than P_n particularly for values of c furthest from zero; however, the differences cannot be said to be large.

As mentioned in section 2 another important test statistic, and one that is widely used in practice, is the t-ratio of the parameter β_0 in the ADF regression (5). The moments of the limiting distribution of the t-ratio, given by $N_{c,\bar{c}}/\sqrt{D_{c,\bar{c}}}$, can be computed using

$$E\left(\frac{N_{c,\bar{c}}^k}{D_{c,\bar{c}}^b}\right) = \frac{1}{\Gamma(b)} \int_0^\infty \theta_2^{b-1} \left.\frac{\partial^k Q(\theta_1, -\theta_2)}{\partial \theta_1^k}\right|_{\theta_1=0} d\theta_2; \tag{12}$$

see Meng (2005, Lemma 1). Obviously, when $b = k$ is integer, this expression coincides with (10). The following integrals define the first two moments of interest.

Theorem 4. *Let $N_{c,\bar{c}}$ and $D_{c,\bar{c}}$ be defined as in (6). Then the first two moments of the ratio $N_{c,\bar{c}}/D_{c,\bar{c}}^{1/2}$ are given by*

$$E\left(\frac{N_{c,\bar{c}}}{D_{c,\bar{c}}^{1/2}}\right) = -(I_1^* + I_2^*), \quad E\left(\frac{N_{c,\bar{c}}^2}{D_{c,\bar{c}}}\right) = I_3^* + I_4^* + I_5^*,$$

where

$$I_1^* = \frac{1}{2} \exp\left(-\frac{c}{2}\right) \int_0^\infty \theta_2^{-1/2} \frac{1}{p(\theta_2)^{1/2}} d\theta_2,$$

$$I_2^* = \frac{1}{2} \exp\left(-\frac{c}{2}\right) \int_0^\infty \theta_2^{-1/2} \frac{q(\theta_2)}{p(\theta_2)^{3/2}} d\theta_2,$$

$$I_3^* = I_2, \; I_4^* = (1/2) I_1,$$

$$I_5^* = \frac{3}{4} \exp\left(-\frac{c}{2}\right) \int_0^\infty \frac{q(\theta_2)^2}{p(\theta_2)^{5/2}} d\theta_2,$$

and the functions $p(\theta_2)$ and $q(\theta_2)$, and integrals I_1 and I_2, are defined in Theorem 3.

Note that numerical computation of the first moment has to deal with an additional singularity at the origin introduced by the component $\theta_2^{-1/2}$. The means and variances of $N_{c,\bar{c}}/\sqrt{D_{c,\bar{c}}}$ are given in Table 8. It can be seen that the mean remains negative over the range of values of c considered and rises with c while the variance also rises with c apart from a small fall around $c = 0$.

Computation of the PDF and CDF of t_0 are also not as straightforward as in the case for $n(\hat{\beta}_0 - 1)$, the reason being that it is $\sqrt{D_{c,\bar{c}}}$ appearing in the denominator of the distribution rather than $D_{c,\bar{c}}$ itself. If the joint MGF/CF of $N_{c,\bar{c}}$ and $\sqrt{D_{c,\bar{c}}}$ were known then the expression in (11) would apply equally well here for the CDF. Unfortunately, this is a difficult function to derive and the methods used to obtain $Q(\theta_1, \theta_2)$ do not appear to be well-suited to this task due to the presence of the square root term. An alternative approach proceeds in two steps. The first step is to use the CF $\Psi(\theta_1, \theta_2)$ to derive the joint PDF of $N_{c,\bar{c}}$ and $D_{c,\bar{c}}$ using a Fourier inversion of the form

$$h(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\{-i(\theta_1 x + \theta_2 y)\} \Psi(\theta_1, \theta_2) d\theta_1 d\theta_2.$$

This PDF can then be used in the second step to derive the PDF of the ratio $N_{c,\bar{c}}/\sqrt{D_{c,\bar{c}}}$ using the expression

$$\begin{aligned} h(z) &= \int_0^{\infty} \sqrt{y} h(z\sqrt{y}, y) dy \\ &= \frac{1}{4\pi^2} \int_0^{\infty} \sqrt{y} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\{-i(\theta_1 z\sqrt{y} + \theta_2 y)\} \Psi(\theta_1, \theta_2) d\theta_1 d\theta_2 dy; \end{aligned}$$

see, for example, Abadir and Rockinger (1997, p.1221). In the case of no detrending Abadir (1995) has used this type of expression to derive closed form analytical expressions for the relevant PDF and CDF, although in the present case, where the CF is of a rather more complicated form, such an outcome appears not to be feasible. The alternative is then to attempt numerical integration, which for the PDF $h(z)$ requires three-fold integration, while the CDF requires a further integration:

$$\begin{aligned} H(z) &= \Pr(t_0 < z) = \int_{-\infty}^{z} h(w) dw \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{z} \int_0^{\infty} \sqrt{y} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\{-i(\theta_1 w\sqrt{y} + \theta_2 y)\} \Psi(\theta_1, \theta_2) d\theta_1 d\theta_2 dy dw. \end{aligned}$$

Given the nature of the function to be integrated this would appear to be a particularly challenging computation to attempt and great care would need to be given to the potential (in)accuracy of the result.

Appendix A. Proofs

Proof of Theorem 1. The MGF of interest is

$$M(t_1,t_2)=E\left[\exp\left(t_1 V_{c,\bar{c}}(1)^2+t_2\int_0^1 V_{c,\bar{c}}^2\right)\right].$$

Using the dependence of $V_{c,\bar{c}}(r)$ on $W_c(r)$ it is straightforward, but somewhat tedious, to show that

$$\begin{aligned} t_1 V_{c,\bar{c}}(1)^2+t_2\int_0^1 V_{c,\bar{c}}^2 &= \bar{a}_1 W_c(1)^2+2a_2 W_c(1)\int_0^1 sW_c(s)ds \\ &\quad +a_3\left(\int_0^1 sW_c(s)ds\right)^2+t_2\int_0^1 W_c(s)^2 ds, \end{aligned}$$

where $\bar{a}_1=(1-\lambda)^2 t_1+(\lambda^2/3)t_2$, $a_2=-3(1-\lambda)^2 t_1-\lambda^2 t_2$ and $a_3=9(1-\lambda)^2 t_1-3(1-\lambda^2)t_2$. Now consider the auxiliary Ornstein-Uhlenbeck (O-U) process $Y(t)$ given by

$$dY(t)=\gamma Y(t)dt+dW(t),\quad Y(0)=0,$$

and let μ_Y be the measure induced by Y. The measures μ_Y and μ_{W_c}, the measure induced by W_c, are equivalent and, by Girsanov's Theorem (see, for example, Theorem 4.1 of Tanaka, 1996),

$$\frac{d\mu_{W_c}}{d\mu_Y}(x)=\exp\left((c-\gamma)\int_0^1 x(s)dx(s)-\frac{(c^2-\gamma^2)}{2}\int_0^1 x(s)^2 ds\right)$$

is the Radon-Nikodym derivative evaluated at $x(t)$, a random process on $[0,1]$ with $x(0)=0$. The change of measure will be used because, for some functional $f(\cdot)$,

$$E\left(f(W_c)\right)=E\left(f(Y)\frac{d\mu_{W_c}}{d\mu_Y}(Y)\right),$$

which will enable the term involving $\int_0^1 W_c^2$ to be eliminated from the MGF. The expression of interest becomes

$$\begin{aligned} M(t_1,t_2) &= E\left\{\exp\left[\bar{a}_1 Y(1)^2+2a_2 Y(1)\int_0^1 sY(s)ds+a_3\left(\int_0^1 sY(s)ds\right)^2\right.\right. \\ &\quad \left.\left.+t_2\int_0^1 Y(s)^2 ds+(c-\gamma)\int_0^1 Y(s)dY(s)-\frac{(c^2-\gamma^2)}{2}\int_0^1 Y(s)^2 ds\right]\right\}. \end{aligned}$$

But $\int_0^1 Y(s)dY(s)=(1/2)[Y(1)^2-1]$; making this substitution yields

$$\begin{aligned} M(t_1,t_2) &= \exp\left(-\frac{(c-\gamma)}{2}\right)E\left\{\exp\left[a_1 Y(1)^2+2a_2 Y(1)\int_0^1 sY(s)ds\right.\right. \\ &\quad \left.\left.+a_3\left(\int_0^1 sY(s)ds\right)^2+\frac{t_2-(c^2-\gamma^2)}{2}\int_0^1 Y(s)^2 ds\right]\right\}. \end{aligned}$$

But the parameter γ is arbitrary, and so we can set $\gamma=\sqrt{c^2-2\theta_2}$ so as to eliminate the term $\int_0^1 Y^2$, thereby obtaining

$$M(t_1,t_2)=\exp\left(-\frac{(c-\gamma)}{2}\right)E\left[\exp\left(w'Aw\right)\right],$$

where

$$w = \left(\begin{array}{c} Y(1) \\ \int_0^1 sY(s)ds \end{array} \right), \quad A = \left(\begin{array}{cc} a_1 & a_2 \\ a_2 & a_3 \end{array} \right), \quad a_1 = \bar{a}_1 + \frac{c-\gamma}{2}.$$

Lemma B1 establishes that $w \sim N(0, \Omega)$ and so it follows that

$$E\left[\exp(w'Aw)\right] = |I_2 - 2\Omega A|^{-1/2}$$

where I_2 is the 2×2 identity matrix and $|\cdot|$ denotes the determinant of a matrix. Then

$$M(t_1, t_2) = \exp\left(-\frac{c}{2}\right) H(t_1, t_2)^{-1/2},$$

where $H(t_1, t_2) = \exp(-\gamma)|I_2 - 2\Omega A|$. Some algebra establishes that

$$|I_2 - 2\Omega A| = 1 - 2a_1\omega^2 - 4a_2\rho - 2a_3 s^2 + 4a_4|\Omega|,$$

where $a_4 = |A|$ and ω^2, ρ and s^2 are the elements of Ω whose definitions can be found in Lemma B1. Taking the product of $e^{-\gamma}$ and $|I_2 - 2\Omega A|$ yields, after some manipulations, the expression for $M(t_1, t_2)$ in the Theorem. □

Proof of Corollary to Theorem 1. First note that

$$M_1(t_1) = M(t_1, 0) = e^{-c/2} H(t_1, 0)^{-1/2}$$

and that $\gamma = c$ when $t_2 = 0$. Furthermore, $H(t_1, 0) = h_1(t_1, 0)\sinh c + h_2(t_1, 0)\cosh c$ while $a_i(t_1, 0) = a_{i1}t_1$ $(i = 1, 2, 3)$ and $a_4(t_1, 0) = 0$. Letting h_{ij}^c $(i = 1, 2; j = 1, 2, 3)$ denote the corresponding h_{ij} coefficients evaluated at $t_2 = 0$ (and hence at $\gamma = c$) it follows that

$$h_i(t_1, 0) = (-1)^i + (h_{i1}^c a_{11} + h_{i2}^c a_{21} + h_{i3}^c a_{31})t_1 = (-1)^i + k_i t_1, \quad i = 1, 2.$$

The result for $M_1(t_1)$ follows by substituting the above expressions into $H(t_1, 0)$ and noting that $\cosh c - \sinh c = e^{-c}$. The derivation of $M_2(t_2)$ follows in a similar fashion by noting that

$$M_2(t_2) = M(0, t_2) = e^{-c/2} H(0, t_2)^{-1/2}$$

and that $H(0, t_2) = h_1(0, t_2)\sinh\gamma + h_2(0, t_2)\cosh\gamma$. It is possible to show that $h_1(0, t_2) = -1 + k_{10} + k_{11}t_2 + k_{12}t_2^2$ and $h_2(0, t_2) = 1 + k_{21}t_2 + k_{22}t_2^2$. The result follows by substition and noting that $\cosh\gamma - \sinh\gamma = e^{-\gamma}$. □

Proof of Theorem 2. First note that

$$m(t) = M\left((1-\bar{c})t, \bar{c}^2 t\right) = \exp\left(-\frac{c}{2}\right) H((1-\bar{c})t, \bar{c}^2 t)^{-1/2}$$

and so

$$\frac{dm(t)}{dt} = -\frac{1}{2}\exp\left(-\frac{c}{2}\right) H((1-\bar{c})t, \bar{c}^2 t)^{-3/2} \frac{dH((1-\bar{c})t, \bar{c}^2 t)}{dt},$$

which needs to be evaluated at $t = 0$. From Theorem 1

$$H(t_1, t_2) = h_1(t_1, t_2)\sinh\gamma + h_2(t_1, t_2)\cosh\gamma,$$

where $\gamma = \sqrt{c^2 - 2t_2}$. The condition $t = 0$ equates to $t_1 = t_2 = 0$ and so immediately we find that $\gamma = c$ in this case. The quantities h_1 and h_2 are linear functions of $a_1, \ldots, a_4$, all of which are zero when $t = 0$ and $\gamma = c$ and so it follows that $h_1(0, 0) = -1$ and $h_2, (0, 0) = 1$.

Combining these results yields

$H(0,0) = h_1(0,0)\sinh c + h_2(0,0)\cosh c = \cosh c - \sinh c = \exp(-c).$

Differentiating $H(\cdot,\cdot)$ with respect to t we find that

$$\frac{dH((1-\bar{c})t,\bar{c}^2t)}{dt} = \frac{dh_1}{dt}\sinh\gamma + h_1\frac{d\sinh\gamma}{dt} + \frac{dh_2}{dt}\cosh\gamma + h_2\frac{d\cosh\gamma}{dt}.$$

For $i = 1,2$ we have

$$\frac{dh_i}{dt} = \frac{d}{dt}\sum_{j=1}^{4} h_{ij}a_j = \sum_{j=1}^{4}\left\{\frac{dh_{ij}}{dt}a_j + h_{ij}\frac{da_j}{dt}\right\}.$$

But, as the $a_j = 0$ when $t = 0$ we can ignore the first components and therefore concentrate on their derivatives. Note that

$$a_j((1-\bar{c})t,\bar{c}^2t) = a_{j0} + a_{j1}(1-\bar{c})t + a_{j2}\bar{c}^2t$$

and so

$$\frac{da_j}{dt} = \frac{da_{j0}}{dt} + a_{j1}(1-\bar{c}) + a_{j2}\bar{c}^2.$$

Now $a_{20} = a_{30} = 0$ while $a_{10} = (c-\gamma)/2$ and so

$$\frac{da_{10}}{dt} = -\frac{1}{2}\frac{d\gamma}{dt} = \frac{\bar{c}^2}{2\gamma}$$

in view of $d\gamma/dt = -\bar{c}^2/\gamma$. It then follows that

$$\begin{aligned}
\frac{da_1}{dt} &= \frac{\bar{c}^2}{2\gamma} + (1-\bar{c})(1-\lambda)^2 + \frac{\bar{c}^2\lambda^2}{3},\\
\frac{da_2}{dt} &= -3(1-\bar{c})(1-\lambda)^2 - \bar{c}^2\lambda^2,\\
\frac{da_3}{dt} &= 9(1-\bar{c})(1-\lambda)^2 - 3\bar{c}^2(1-\lambda^2),\\
\frac{da_4}{dt} &= \frac{da_1}{dt}a_3 + a_1 2\frac{da_3}{dt} - 2a_2\frac{da_2}{dt} = 0.
\end{aligned}$$

Evaluating the h_{ij} at $\gamma = c$ and using the above results yields (after some simplification)

$$\begin{aligned}
\frac{dh_1}{dt} &= -\frac{2}{c}\left(\frac{\bar{c}^2}{2c} + (1-\bar{c})(1-\lambda)^2 + \frac{\bar{c}^2\lambda^2}{3}\right) - \frac{4}{c^3}\left(3(1-\bar{c})(1-\lambda)^2 + \bar{c}^2\lambda^2\right)\\
&\quad +6\left(\frac{1}{3c^2} + \frac{1}{c^4} - \frac{1}{c^5}\right)\left(3(1-\bar{c})(1-\lambda)^2 - \bar{c}^2(1-\lambda^2)\right),
\end{aligned}$$

$$\begin{aligned}
\frac{dh_2}{dt} &= \frac{2}{c^2}\left(3(1-\bar{c})(1-\lambda)^2 + \bar{c}^2(1+\lambda^2)\right)\\
&\quad -6\left(\frac{1}{c^3} - \frac{1}{c^4}\right)\left(3(1-\bar{c})(1-\lambda)^2 - \bar{c}^2(1-\lambda^2)\right).
\end{aligned}$$

We also need

$$\frac{d\sinh\gamma}{dt} = \frac{d\sinh\gamma}{d\gamma}\frac{d\gamma}{dt} = -\frac{\bar{c}^2}{c}\cosh\gamma,$$

$$\frac{d\cosh\gamma}{dt} = \frac{d\cosh\gamma}{d\gamma}\frac{d\gamma}{dt} = -\frac{\bar{c}^2}{c}\sinh\gamma,$$

both evaluated at $t=0$, to obtain

$$\left.\frac{dH}{dt}\right|_{t=0} = \frac{dh_1}{dt}\sinh c + \frac{\bar{c}^2}{c}\cosh c + \frac{dh_2}{dt}\cosh c - \frac{\bar{c}^2}{c}\sinh c$$
$$= \frac{\bar{c}^2}{c}e^{-c} + \eta_1 \sinh c + \eta_2 \cosh c,$$

where $\eta_i = dh_i/dt$ $(i=1,2)$; the form of the η_i given in Table 2 is derived from the expressions given earlier in terms of inverse powers of c. It then follows that

$$E(S_{c,\bar{c}}) = -\frac{1}{2}e^{-c/2}(e^{-c})^{-3/2}\left.\frac{dH}{dt}\right|_{t=0} = -\frac{1}{2}e^{c}\left(\frac{\bar{c}^2}{c}e^{-c} + \eta_1 \sinh c + \eta_2 \cosh c\right)$$

as required (for $c \neq 0$). Care has to be taken when $c=0$ because the η_i are expressed in terms of inverse powers of c. However, closer inspection of the terms of the products $\eta_1 \sinh c$ and $\eta_2 \cosh c$, allied with the expansions

$$\sinh c = c + \frac{c^3}{3} + \frac{c^5}{5} + \ldots, \quad \cosh c = 1 + \frac{c^2}{2} + \frac{c^4}{4} + \ldots,$$

yields the expression stated in the Theorem.

Turning to the second moment, a further differentiation of $m(t)$ yields

$$\frac{d^2 m(t)}{dt^2} = \frac{3}{4}e^{-c/2}H((1-\bar{c})t, \bar{c}^2 t)^{-5/2}\left(\frac{dH((1-\bar{c})t, \bar{c}^2 t)}{dt}\right)^2$$
$$-\frac{1}{2}e^{-c/2}H((1-\bar{c})t, \bar{c}^2 t)^{-3/2}\frac{d^2 H((1-\bar{c})t, \bar{c}^2 t)}{dt^2}.$$

The second derivative of $H(\cdot,\cdot)$ is given by

$$\frac{d^2 H}{dt^2} = \frac{d^2 h_1}{dt^2}\sinh\gamma + 2\frac{dh_1}{dt}\frac{d\sinh\gamma}{dt} + h_1\frac{d^2\sinh\gamma}{dt^2}$$
$$+\frac{d^2 h_2}{dt^2}\cosh\gamma + 2\frac{dh_2}{dt}\frac{d\cosh\gamma}{dt} + h_2\frac{d^2\cosh\gamma}{dt^2}.$$

When $t=0$ the second derivatives of $\sinh\gamma$ and $\cosh\gamma$ are equal to

$$\frac{d^2\sinh\gamma}{dt^2} = \frac{\bar{c}^4}{c^2}\left(\sinh c - \frac{1}{c}\cosh c\right), \quad \frac{d^2\cosh\gamma}{dt^2} = \frac{\bar{c}^4}{c^2}\left(\cosh c - \frac{1}{c}\sinh c\right),$$

while the second derivatives of h_1 and h_2 take the form

$$\frac{d^2 h_i}{dt^2} = \sum_{j=1}^{4}\left(\frac{d^2 h_{ij}}{dt^2}a_j + 2\frac{dh_{ij}}{dt}\frac{da_j}{dt} + h_{ij}\frac{d^2 a_j}{dt^2}\right), \quad (i=1,2).$$

Calculation of the appropriate derivatives ultimately results in

$$\frac{d^2 h_1}{dt^2} = -3\frac{\bar{c}^4}{c^4} + 8\bar{c}^4\lambda^4\left(\frac{1}{3c^3} + \frac{1}{c^5}\right) - 4\bar{c}^2\left(\frac{1}{c^3} + \frac{8}{c^5}\right)\left(3(1-\bar{c})(1-\lambda)^2 + \bar{c}^2\lambda^2\right)$$
$$+12\bar{c}^2\left(\frac{1}{c^4} + \frac{5}{c^6} - \frac{5}{c^7}\right)\left(3(1-\bar{c})(1-\lambda)^2 - \bar{c}^2(1-\lambda^2)\right)$$

$$\frac{d^2 h_2}{dt^2} = 16\frac{\bar{c}^2}{c^4}\left(3(1-\bar{c})(1-\lambda)^2 + \bar{c}^2\lambda^2 + \frac{1}{2}\bar{c}^2(1-\lambda^4)\right)$$
$$-48\bar{c}^2\left(\frac{1}{c^5} - \frac{1}{c^4}\right)\left(3(1-\bar{c})(1-\lambda)^2 - \bar{c}^2(1-\lambda^2)\right).$$

Combining all these results and evaluating at $t = 0$ yields the expression in the Theorem for $c \neq 0$. The expression for $c = 0$ is obtained by analysing appropriate expansions in c and showing that all those with negative powers cancel out. $\square$

Proof of Theorem 3. From the definition of $Q(\theta_1, \theta_2)$ we obtain

$$\begin{aligned}\frac{\partial Q(\theta_1,\theta_2)}{\partial\theta_1} &= -\frac{1}{2}\exp\left(-\frac{\theta_1+c}{2}\right)H(\theta_1,\theta_2)^{-1/2}\\ &\quad -\frac{1}{2}\exp\left(-\frac{\theta_1+c}{2}\right)H(\theta_1,\theta_2)^{-3/2}\frac{\partial H(\theta_1,\theta_2)}{\partial\theta_1}.\end{aligned}$$

Partial differentiation of $H(\theta_1, \theta_2)$ yields

$$\frac{\partial H(\theta_1,\theta_2)}{\partial\theta_1} = \frac{\partial h_1}{\partial\theta_1}\sinh\gamma + \frac{\partial h_2}{\partial\theta_1}\cosh\gamma$$

where $\partial h_i/\partial\theta_1 = \sum_{j=1}^4 h_{ij}\partial a_j(\theta_1,\theta_2)/\partial\theta_1$. Hence

$$\left.\frac{\partial H(\theta_1,\theta_2)}{\partial\theta_1}\right|_{\theta_1=0} = q_1(\theta_2)\sinh\sqrt{c^2+2\theta_2} + q_2(\theta_2)\cosh\sqrt{c^2+2\theta_2}$$

where the q_i are defined in the Theorem. Also

$$\begin{aligned}H(0,-\theta_2) &= h_1(0,-\theta_2)\sinh\sqrt{c^2+2\theta_2} + h_2(0,-\theta_2)\cosh\sqrt{c^2+2\theta_2}\\ &= p_1(\theta_2)\sinh\sqrt{c^2+2\theta_2} + p_2(\theta_2)\cosh\sqrt{c^2+2\theta_2}\end{aligned}$$

where the definition of the p_i is obvious. Setting $k = 1$ in (10) and using the above expressions yields the result for the first moment.

Turning to the second moment, setting $k = 2$ in (10), we need to find

$$\begin{aligned}\frac{\partial^2 Q(\theta_1,\theta_2)}{\partial\theta_1^2} &= \frac{1}{4}\exp\left(-\frac{\theta_1+c}{2}\right)H(\theta_1,\theta_2)^{-1/2}\\ &\quad +\frac{1}{2}\exp\left(-\frac{\theta_1+c}{2}\right)H(\theta_1,\theta_2)^{-3/2}\frac{\partial H(\theta_1,\theta_2)}{\partial\theta_1}\\ &\quad +\frac{2}{4}\exp\left(-\frac{\theta_1+c}{2}\right)H(\theta_1,\theta_2)^{-5/2}\left[\frac{\partial H(\theta_1,\theta_2)}{\partial\theta_1}\right]^2\\ &\quad -\frac{1}{2}\exp\left(-\frac{\theta_1+c}{2}\right)H(\theta_1,\theta_2)^{-3/2}\frac{\partial^2 H(\theta_1,\theta_2)}{\partial\theta_1^2}.\end{aligned}$$

The components of the first three terms have been derived above, so we therefore need

$$\frac{\partial^2 H(\theta_1,\theta_2)}{\partial\theta_1^2} = \frac{\partial^2 h_1}{\partial\theta_1^2}\sinh\gamma + \frac{\partial^2 h_2}{\partial\theta_1^2}\cosh\gamma = 0$$

upon inspection of the relevant derivatives when evaluated at $\theta_1 = 0$ and $\theta_2 = -\theta_2$. The second moment is, therefore, the sum of the integrals of the first three terms which are defined in the Theorem. $\square$

Proof of Theorem 4. This follows straightforwardly from (12) using the derivatives derived in the proof of Theorem 3. $\square$

Appendix B. Supplementary result

Lemma B1. *Let $Y(t)$ satisfy $dY(t) = \gamma Y(t)dt + dW(t)$, where $W(t)$ is a Wiener process on $C[0,1]$ and $Y(0) = 0$. Then the vector*

$$w = \begin{pmatrix} Y(1) \\ \int_0^1 sY(s)ds \end{pmatrix} \sim N(0, \Omega),$$

where

$$\Omega = \begin{pmatrix} \omega^2 & \rho \\ \rho & s^2 \end{pmatrix}$$

and its elements are defined by

$$\omega^2 = \frac{e^{2\gamma} - 1}{2\gamma}, \quad \rho = \frac{e^{2\gamma}}{2\gamma}\left(\frac{1}{\gamma} - \frac{1}{\gamma^2}\right) + \left(\frac{1}{2\gamma^2} + \frac{1}{2\gamma^3}\right),$$

$$s^2 = e^{2\gamma}\left(\frac{1}{2\gamma^3} - \frac{1}{\gamma^4} + \frac{1}{2\gamma^5}\right) + \frac{1}{3\gamma^2} + \frac{1}{2\gamma^3} - \frac{1}{2\gamma^5}.$$

Proof. The process $Y(t)$ has the solution $Y(t) = \int_0^t e^{(t-r)\gamma} dW(r)$ and hence $E[Y(t)] = 0$ for all t. Setting $t = 1$ it then follows that

$$\omega^2 = E[Y(1)^2] = E\left(\int_0^1 e^{(1-r)\gamma} dW(r)\right)^2 = \int_0^1 e^{2(1-r)\gamma} dr = \frac{e^{2\gamma} - 1}{2\gamma}$$

as required. From the above solution we obtain

$$\int_0^1 sY(s)ds = \int_0^1 s \int_0^s e^{(s-r)\gamma} dW(r)ds = \int_r^1 s \int_0^1 e^{(s-r)\gamma} dW(r)ds = \int_0^1 v(r)dW(r),$$

where $v(r) = e^{-r\gamma} \int_r^1 se^{s\gamma} ds$. Clearly $E[tY(t)] = 0$ while the variance of $\int_0^1 sY(s)ds$ is equal to $s^2 = \int_0^1 v(r)^2 dr$. Some tedious algebra establishes that s^2 has the stated form. Finally we need an expression for

$$\rho = E\left[Y(1) \int_0^1 sY(s)ds\right] = \int_0^1 E\left[Y(1)sY(s)\right] ds.$$

For $t < 1$ we have $tY(t)Y(1) = t \int_0^t e^{(t-r)\gamma} dW(r) \int_0^1 e^{(1-s)\gamma} dW(s)$ and so

$$\begin{aligned} E\left[tY(t)Y(1)\right] &= E\left[t \int_0^t e^{(t-r)\gamma} dW(r) \left(\int_0^t e^{(1-s)\gamma} dW(s) + \int_t^1 e^{(1-s)\gamma} dW(s)\right)\right] \\ &= t \int_0^t e^{(t+1-2r)\gamma} dr = \frac{e^\gamma}{\gamma} t \sinh t\gamma. \end{aligned}$$

The required integral is therefore $\int_0^1 t \sinh t\gamma dt$. Using (2.473.1) of Gradshteyn and Ryzhik (1994) we find that

$$\int_0^1 t \sinh t\gamma dt = \frac{\cosh \gamma}{\gamma} - \frac{\sinh \gamma}{\gamma^2}$$

and hence $\rho = (e^\gamma/\gamma^2)(\cosh \gamma - \gamma^{-1} \sinh \gamma)$ which can also be written in the form in the Lemma by recalling that $\cosh \gamma = (e^\gamma + e^{-\gamma})/2$ and $\sinh \gamma = (e^\gamma - e^{-\gamma})/2$. □

Appendix C. Computational details

Computations based on the CF $\Phi(t_1, t_2)$ and other related functions involve the square root of the complex-valued function $H(it_1, it_2)$ which is defined in Theorem 1. Care must be taken when computing such square roots as most software computes the principal value that can lead to discontinuities in the function. The approach adopted here to ensure continuity of the real and imaginary parts of the square root function follows the method outlined in Tanaka (1996, p.183) which proceeds by first computing the CF at the origin and then checking the bahaviour of the function when avaluated at successive small increments. An alternative (but essentially equivalent) method was used by Perron (1989).

The integrals used to compute the first two moments of the ratio $N_{c,\bar{c}}/D_{c,\bar{c}}$ in Theorem 3 and of $N_{c,\bar{c}}/D_{c,\bar{c}}^{1/2}$ in Theorem 4 were computed using the change of variable $x = (c^2 + 2\theta_2)^{1/2}$. For example, the integral I_1 becomes

$$I_1 = \frac{1}{2} \exp\left(-\frac{c}{2}\right) \int_c^\infty \frac{x}{p((x^2 - c^2)/2)^{1/2}} dx$$

where $p((x^2 - c^2)/2) = p_1((x^2 - c^2)/2) \sinh x + p_2((x^2 - c^2)/2) \cosh x$. The upper limit for these integrals was chosen as the value of x for which the modulus of the integrand was less than 1×10^{-8}. For the integrals used to compute CDFs and PDFs the range of integration was taken as $[\epsilon, \bar{U}]$ with $\epsilon = 1 \times 10^{-8}$ and $\bar{U}$ determined as in Perron (1989, p.254). All numerical integration was carried out using Romberg's method which, as a by-product, enables a measure of accuracy of the final value to be determined from the last step in the approximation. For example, the largest absolute error of the integrals used to construct Table 5 was 4.16×10^{-10}. As a further accuracy check the integrals were also computed using Simpson's method and the results were verified.

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Table 1
The coefficients in Theorem 1 and its Corollary

i	h_{i1}	h_{i2}	h_{i3}	h_{i4}
1	$-\dfrac{2}{\gamma}$	$\dfrac{4}{\gamma^3}$	$2\left(\dfrac{1}{3\gamma^2}+\dfrac{1}{\gamma^4}-\dfrac{1}{\gamma^5}\right)$	$4\left(\dfrac{1}{3\gamma^3}+\dfrac{1}{\gamma^5}\right)$
2	0	$-\dfrac{4}{\gamma^2}$	$-2\left(\dfrac{1}{3\gamma^2}+\dfrac{1}{\gamma^3}-\dfrac{1}{\gamma^4}\right)$	$-\dfrac{4}{\gamma^4}$
i	a_{i0}	a_{i1}	a_{i2}	
1	$\frac{1}{2}(c-\gamma)$	$(1-\lambda)^2$	$\frac{1}{3}\lambda^2$	
2	0	$-3(1-\lambda)^2$	$-\lambda^2$	
3	0	$9(1-\lambda)^2$	$-3(1-\lambda^2)$	
i	h^c_{i1}	h^c_{i2}	h^c_{i3}	
1	$-\dfrac{2}{c}$	$\dfrac{4}{c^3}$	$2\left(\dfrac{1}{3c^2}+\dfrac{1}{c^4}-\dfrac{1}{c^5}\right)$	
2	0	$-\dfrac{4}{c^2}$	$-2\left(\dfrac{1}{3c^2}+\dfrac{1}{c^3}-\dfrac{1}{c^4}\right)$	

Table 2
The coefficients in Theorem 2

j	η_{1j}	η_{2j}	η_{3j}	η_{4j}
1	$-\frac{2}{3}g_1$	0	$-4\bar{c}^2\left(g_1-\frac{2}{3}\bar{c}^2\lambda^4\right)$	0
2	$2\left(g_2-\frac{1}{2}\bar{c}^2\right)$	$2g_3$	0	0
3	$-4g_1$	$-6g_2$	$-4\bar{c}^2\left(g_1-\frac{2}{3}\bar{c}^2\lambda^4\right)$	0
4	$6g_2$	$6g_2$	$12\bar{c}^2\left(g_2-\frac{1}{4}\bar{c}^2\right)$	$16\bar{c}^2\left(g_1+\frac{1}{2}\bar{c}^2(1-\lambda^2)\right)$
5	$-6g_2$	0	$-32\bar{c}^2\left(g_1-\frac{1}{4}\bar{c}^2\lambda^4\right)$	$-48\bar{c}^2g_2$
6	0	0	$60\bar{c}^2g_2$	$48\bar{c}^2g_2$
7	0	0	$-60\bar{c}^2g_2$	0

Note: $g_1 = 3(1-\bar{c})(1-\lambda)^2 + \bar{c}^2\lambda^2$, $g_2 = 3(1-\bar{c})(1-\lambda)^2 - \bar{c}^2(1-\lambda^2)$ and $g_3 = 3(1-\bar{c})(1-\lambda)^2 + \bar{c}^2(1+\lambda^2)$.

Table 3
Means and variances of $S_{c,\bar{c}}$ for $\bar{c} = -13.5$

c	Mean	Variance
−20.0	4.3113	1.8740
−10.0	7.6733	10.9986
−5.0	12.2591	48.3724
−2.0	17.6668	154.4212
−1.0	19.6251	214.1051
−0.5	20.3058	237.3413
0.0	20.5662	245.4562
0.5	20.3166	233.4255
1.0	20.0035	217.2866
2.0	34.9408	1196.8373

Table 4
Percentage points of $S_{c,\bar{c}}$ for $\bar{c} = -13.5$

c	0.01	0.05	0.10	0.50	0.90	0.95	0.99
−20.0	2.0289	2.4780	2.7633	4.1015	6.1273	6.8608	8.4666
−10.0	2.7698	3.5967	4.1622	7.0126	12.0214	14.0193	18.5015
−5.0	3.3943	4.6316	5.4914	10.5484	21.1692	25.7770	37.2201
−2.0	3.8232	5.3954	6.5555	14.1492	33.1657	42.1791	66.1949
−1.0	3.9305	5.6011	6.8544	15.3280	37.7705	48.6688	78.3244
−0.5	3.9635	5.6660	6.9499	15.7249	39.4042	50.9817	82.7480
0.0	3.9756	5.6900	6.9853	15.8750	40.0304	51.8768	84.4433
0.5	3.9640	5.6669	6.9513	15.7310	39.4304	51.0193	82.8208
1.0	3.9476	5.6353	6.9052	15.5455	38.6855	49.9640	80.8151
2.0	4.2962	6.4318	8.1519	22.7660	77.8535	106.9800	208.8539

Table 5
Means and variances of $N_{c,\bar{c}}/D_{c,\bar{c}}$ for $\bar{c} = -13.5$

c	Mean	Variance
−20.0	−24.1340	56.8631
−10.0	−14.4969	38.2772
−5.0	−9.9330	29.9027
−2.0	−7.6137	26.0755
−1.0	−7.0848	25.3166
−0.5	−6.9238	25.1033
0.0	−6.8648	25.0286
0.5	−6.9213	25.1021
1.0	−6.9929	25.2712
2.0	−4.8549	25.4507
3.0	−0.6006	18.8752
4.0	1.6832	9.4307
5.0	2.8293	4.2145

Table 6
Percentage points of $N_{c,\bar{c}}/D_{c,\bar{c}}$ for $\bar{c} = -13.5$

c	0.01	0.05	0.10	0.50	0.90	0.95	0.99
−20.0	−46.3421	−38.0513	−34.1640	−23.0867	−15.4533	−13.7906	−11.1524
−10.0	−33.6978	−26.1695	−22.7289	−13.4120	−7.6697	−6.5295	−4.8175
−5.0	−27.6011	−20.4199	−17.1956	−8.8069	−4.1406	−3.2949	−2.0746
−2.0	−24.5493	−17.5098	−14.3820	−6.4524	−2.3766	−1.6871	−0.6995
−1.0	−23.8758	−16.8604	−13.7504	−5.9136	−1.9676	−1.3112	−0.3738
−0.5	−23.6744	−16.6653	−13.5602	−5.7495	−1.8410	−1.1940	−0.2718
0.0	−23.6014	−16.5945	−13.4910	−5.6895	−1.7941	−1.1505	−0.2337
0.5	−23.6716	−16.6626	−13.5575	−5.7470	−1.8387	−1.1916	−0.2694
1.0	−23.7737	−16.7601	−13.6517	−5.8222	−1.8822	−1.2251	−0.2855
2.0	−21.7352	−14.6888	−11.5655	−3.6464	0.2361	0.7907	1.5616

Table 7
Asymptotic power of P_n and $n(\hat{\beta}_0 - 1)$

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
c	P_n	$n(\hat{\beta}_0 - 1)$	P_n	$n(\hat{\beta}_0 - 1)$	P_n	$n(\hat{\beta}_0 - 1)$
−20	0.4602	0.4717	0.8519	0.8542	0.9557	0.9568
−19	0.4082	0.4187	0.8126	0.8159	0.9370	0.9380
−18	0.3579	0.3673	0.7675	0.7689	0.9126	0.9133
−17	0.3101	0.3185	0.7170	0.7178	0.8817	0.8821
−16	0.2654	0.2728	0.6618	0.6622	0.8440	0.8439
−15	0.2245	0.2309	0.6032	0.6031	0.7991	0.7985
−14	0.1876	0.1930	0.5424	0.5419	0.7475	0.7465
−13	0.1549	0.1595	0.4810	0.4802	0.6900	0.6885
−12	0.1257	0.1303	0.4217	0.4195	0.6272	0.6260
−11	0.1017	0.1053	0.3631	0.3613	0.5623	0.5607
−10	0.0814	0.0842	0.3082	0.3068	0.4963	0.4945
−9	0.0645	0.0666	0.2581	0.2569	0.4312	0.4294
−8	0.0507	0.0524	0.2133	0.2125	0.3690	0.3674
−7	0.0396	0.0409	0.1743	0.1738	0.3113	0.3099
−6	0.0308	0.0318	0.1412	0.1408	0.2594	0.2583
−5	0.0240	0.0247	0.1136	0.1135	0.2141	0.2133
−4	0.0188	0.0193	0.0914	0.0914	0.1759	0.1754
−3	0.0150	0.0153	0.0741	0.0741	0.1450	0.1447
−2	0.0123	0.0124	0.0614	0.0614	0.1215	0.1214
−1	0.0106	0.0107	0.0531	0.0531	0.1059	0.1059
0	0.0100	0.0100	0.0500	0.0500	0.1000	0.1000

Table 8
Means and variances of $N_{c,\bar{c}}/\sqrt{D_{c,\bar{c}}}$ for $\bar{c} = -13.5$

c	Mean	Variance
−20.0	−3.3975	0.3499
−10.0	−2.5898	0.3538
−5.0	−2.0913	0.3889
−2.0	−1.7717	0.4489
−1.0	−1.6867	0.4745
−0.5	−1.6597	0.4832
0.0	−1.6559	0.4659
0.5	−1.6592	0.4836
1.0	−1.6690	0.4848
2.0	−1.1582	0.9326

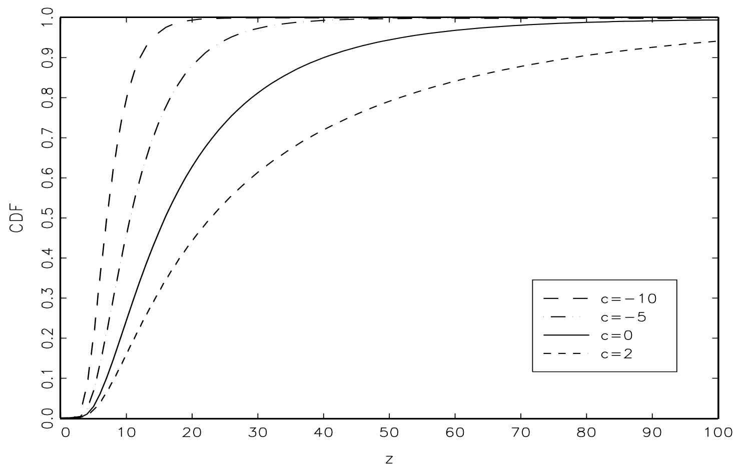

Figure 1. CDF of Limit Distribution of P_n

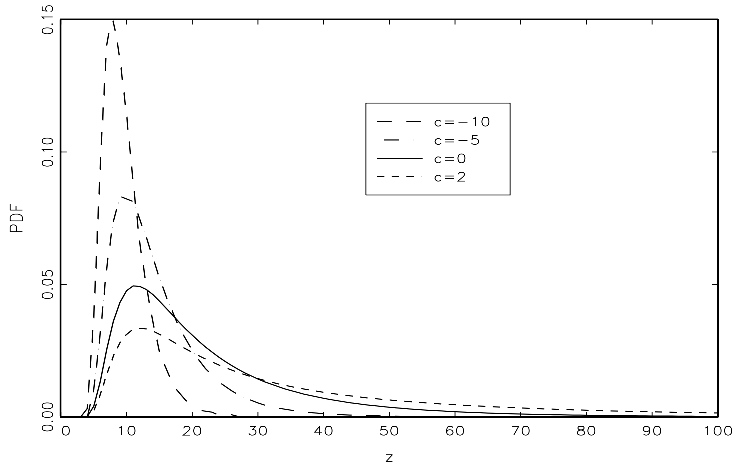

Figure 2. PDF of Limit Distribution of P_n

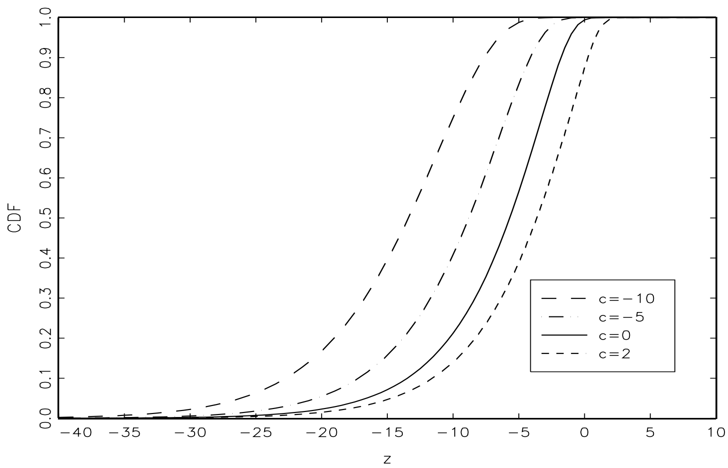

Figure 3. CDF of Limit Distribution of $n(\hat{\beta}_0 - 1)$

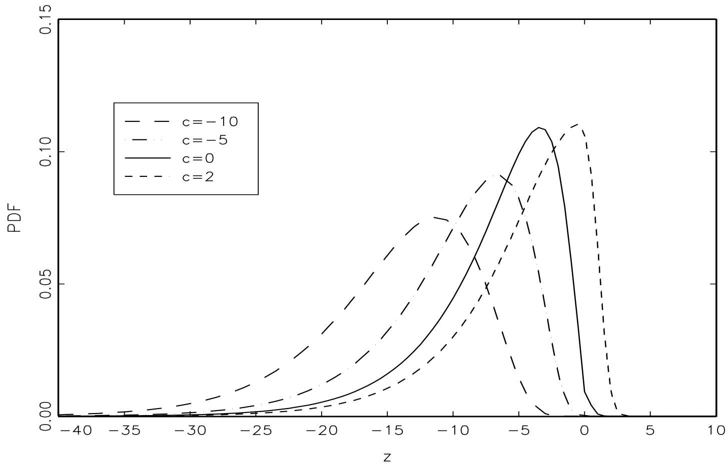

Figure 4. PDF of Limit Distribution of $n(\hat{\beta}_0 - 1)$

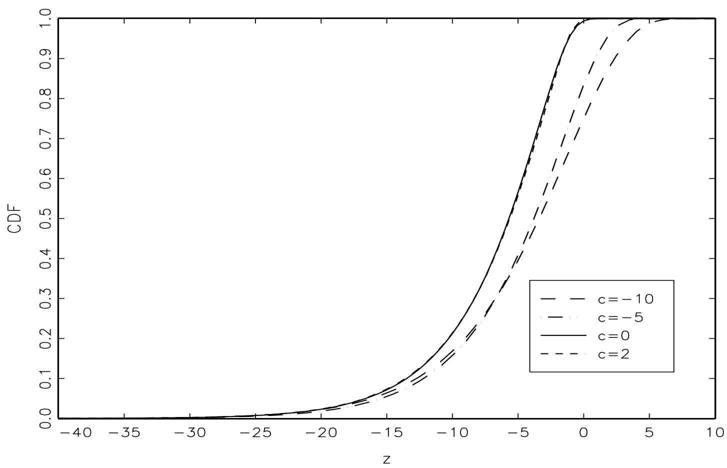

Figure 5. CDF of Limit Distribution of $n(\hat{\beta}_0 - \beta_0)$

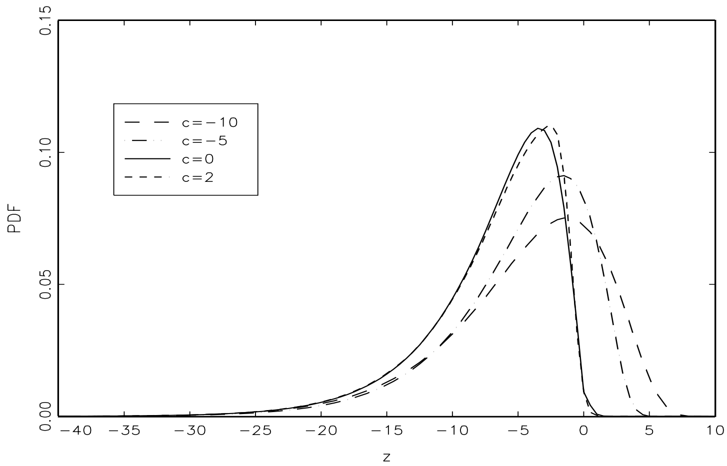

Figure 6. PDF of Limit Distribution of $n(\hat{\beta}_0 - \beta_0)$